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A Bethe ansatz solution for the closed $U_q[sl(2)]$ Temperley–Lieb quantum spin chains

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Abstract. We solve the spectrum of the closed Temperley–Lieb quantum spin chains using the coordinate Bethe ansatz. These models are invariant under the quantum group $U_q[sl(2)]$.

1. Introduction

Quantum groups, together with the Temperley–Lieb algebra, play a important role in the study of integrable spin chains. It may be interesting to study particular Hamiltonians associated with the Temperley–Lieb which are invariant to the quantum group. Taking into account the usual toroidal boundary conditions, the Hamiltonians take the form

$$\mathcal{H} = \sum_{n=1}^{N-1} U_{n,n+1} + U_{N1}. \tag{1}$$

where $U_{n,n+1}$ operates in a direct product of two $(2s+1)$-dimensional complex spaces V^{2s+1} at positions n and $n+1$. They are not invariant with respect to $U_q[sl(2)]$ since $U_{N1} \neq U_{1N}$ breaks translational invariance, reflecting the non-cocommutativity of the co-product. Indeed, we know from [1–3] that very special boundary terms must be considered when we seek these quantum group invariant spin chains. In particular, one possibility to obtain a quantum group invariant Hamiltonian is to consider open boundary conditions, i.e. $U_{N1} = 0$. for the XXZ Hamiltonian with open boundary conditions one has to apply the Bethe ansatz techniques introduced by Sklyanin [4] using Cherednik's reflection matrices [5, 6]. The XXZ Heisenberg model [7], $spl_q(2,1)$ invariant supersymmetric t–J model [8, 9], $U_q[sl(n)]$ invariant generalization of the XXZ chain [10] and $SU_q(n|m)$ spin chains [3, 11] have been solved for open boundary conditions by this method.

Recently, by means of a generalized algebraic nested Bethe ansatz, Karowski and Zapletal [12] presented a class of quantum group invariant n-state vertex models with periodic boundary conditions. Also an extension of this method to the case of graded vertex models was analyzed in [13], where a $spl_q(2|1)$ invariant SUSY t–J model with boundary conditions was presented.

In fact, this type of models were first discussed by Martin [14] from the representations of the Hecke algebra. The study of closed quantum group invariant closed spin chains in the framework of the coordinate Bethe ansatz was presented by Grosse *et al* for the

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$SU_q(2)$ case [15]. In this context it would be interesting to discuss other quantum group invariant closed spin chains. Therefore, it is the purpose of this paper to present and solve, via coordinate Bethe ansatz [16] a closed spin-s Hamiltonian, which in terms of the Temperley–Lieb operators can be written as

$$\mathcal{H} = \sum_{n=1}^{N-1} U_n + \mathcal{U}_0 \tag{2}$$

where

$$\mathcal{U}_0 = G U_{N-1} G^{-1} \qquad G = (Q - U_1)(Q - U_2)\cdots(Q - U_{N-1}) \tag{3}$$

satisfying $[\mathcal{H}, G] = 0$ and additionally invariance with respect to the quantum algebra. The operator G shifts the U_n by one unit $G U_n G^{-1} = U_{n+1}$ and maps $\mathcal{U}_0$ onto U_1, which manifest the translational invariance of $\mathcal{H}$.

2. The Temperley–Lieb Hamiltonians

In the basis where S_n^z is diagonal with eigenvectors $|s, n\rangle, |s-1, n\rangle, \ldots, |-s, n\rangle$ and eigenvalues $s, s-1, \ldots, -s$, the Hamiltonian densities acting on two neighbouring sites are given by

$$\langle k, l|\ U\ |i, j\rangle = \epsilon(i)\epsilon(k) q^{(i+k)} \delta_{i+j,0}\ \delta_{k+l,0} \qquad i, j, k, l = s, s-1, \ldots, -s+1, -s \tag{4}$$

where $\epsilon(i) = (-1)^i$ for s integer and $\epsilon(i) = (-1)^{i+1/2}$ for s semi-integer. Thus U_n denotes the projection on states whose restriction to sites n and $n+1$ has total spin zero. These Hamiltonians were derived Batchelor and Kuniba [17] from representations of the Temperley–Lieb algebras associated with the quantum group $U_q[sl(2)]$. The case $s = \frac{1}{2}$ was investigated in [15].

In fact, U_n obeys the Temperley–Lieb algebra [18]

$$\begin{aligned} &U_n^2 = (Q + Q^{-1}) U_n \qquad Q + Q^{-1} = [2s+1]_q \\ &U_n U_{n+1} U_n = U_n \quad [U_n, U_l] = 0 \qquad \text{for } |n - l| \geqslant 2 \end{aligned} \tag{5}$$

and commutes with the quantum group $U_q[sl(2)]$. The q-number notation is $[x]_q = (q^x - q^{-x})/(q - q^{-1})$. This algebra appears in a large class of solvable models and is known to essentially govern their physical properties: $\mathcal{H}$ is an element of a set of infinity quantities conserved which are involutive provided that U_n satisfies the defining relations (5).

Having now built common ground for all closed Hamiltonian densities, whose salient feature is that they are spin-zero projectors, we may implement the steps of [19], where the spectrum of the A–D Temperley–Lieb Hamiltonians with either periodic or free boundary conditions were solved, via a generalization of the coordinate Bethe ansatz.

3. The coordinate Bethe ansatz

Since these Hamiltonians commute with the total spin $S_T^z = \sum_{n=1}^N S_n^z$, the eigenvalues of the operator $r = sN - S_T^z$ can be used to collect the eigenstates of $\mathcal{H}$ in sectors, Ψ_r. Due to this $U(1)$ invariance, there always exists a reference state Ψ_0 satisfying $\mathcal{H}\Psi_0 = E_0\Psi_0$, with $E_0 = 0$. We take Ψ_0 to be $\Psi_0 = \prod_n |s, n\rangle$. This is the only eigenstate in the sector $r = 0$. All other energies will be measured relative to this state.

We will now start to diagonalize $\mathcal{H}$ in every sector. Nothing interesting happens in sectors with $r < 2s$. Since $\mathcal{H}$ is a sum of projectors on spin zero, these states are annihilated by $\mathcal{H}$.

The first non-trivial sector $r = 2s$, the correspondent eigenspace is spanned by the states

$$|n(-j, j)\rangle = \left| s\ s \cdots s\ \underset{n}{-j\ j}\ s \cdots s \right\rangle$$

where $n = 1, 2, \ldots, N-1$ and $j = -s, -s+1, \ldots, s$. We seek eigenstates of $\mathcal{H}$ which are linear combinations of these vectors. It is very convenient to consider the linear combination

$$|\Omega(n)\rangle = \sum_{j=-s}^{s} (-1)^{s+j}\ q^{s-j}\ |n(-j, j)\rangle\,. \tag{6}$$

which is a highest-weight state, $S^+ |\Omega(n)\rangle = 0$, and eigenstate of U_n:

$$\begin{aligned} &U_n |\Omega(n)\rangle = (Q + Q^{-1}) |\Omega(n)\rangle \qquad U_{n\pm1} |\Omega(n)\rangle = \epsilon_s |\Omega(n \pm 1)\rangle \\ &U_n |\Omega(n \pm 1)\rangle = \epsilon_s |\Omega(n)\rangle \qquad U_n |\Omega(m)\rangle = 0 \qquad \text{for } n \neq \{m \pm 1, m\} \end{aligned} \tag{7}$$

where $\epsilon_s = -1$ for semi-integer s and $\epsilon_s = 1$ for integer s. In this basis, all spin-s Hamiltonians $\mathcal{H}$ can be treated in a similar way and it affords a considerable simplification in the diagonalization of $\mathcal{H}$, when one compares with the computations in the usual spin basis [19].

3.1. One-pseudoparticle eigenstates

Let us consider one free pseudoparticle as a highest-weight state which lies in the sector $r = 2s$:

$$\Psi_{2s} = \sum_{n=1}^{N-1} A(n)\, |\Omega(n)\rangle\,. \tag{8}$$

Using the eigenvalue equation $\mathcal{H}\Psi_{2s} = E_{2s}\Psi_{2s}$, one can derive a complete set of equations for the wavefunctions $A(n)$.

The action of the operator $G = (Q - U_1) \cdots (Q - U_{N-1})$ on the states $|\Omega(n)\rangle$ can be computed using equation (7). It is simple in the bulk and at the left boundary:

$$G\, |\Omega(n)\rangle = -\epsilon_s Q^{N-2}\ |\Omega(n+1)\rangle \qquad 1 \leqslant n \leqslant N-2 \tag{9}$$

but manifests its non-locality at the right boundary:

$$G\, |\Omega(N-1)\rangle = \epsilon_s Q^{N-2} \sum_{n=1}^{N-1} (-\epsilon_s Q)^{-n}\ |\Omega(N-n)\rangle\,. \tag{10}$$

Similarly, acting with the operator $G^{-1} = (Q^{-1} - U_{N-1}) \cdots (Q^{-1} - U_1)$, we obtain

$$G^{-1} |\Omega(n)\rangle = -\epsilon_s Q^{-N+2} |\Omega(n-1)\rangle \qquad 2 \leqslant n \leqslant N-1 \tag{11}$$

$$G^{-1} |\Omega(1)\rangle = \epsilon_s Q^{-N+2} \sum_{n=1}^{N-1} (-\epsilon_s Q)^{n}\ |\Omega(n)\rangle \tag{12}$$

for the bulk including the right boundary and for the left boundary, respectively.

From these results one can see that the action of $\mathcal{U}_0 = G U_{N-1} G^{-1}$ vanishes in the bulk:

$$\mathcal{U}_0 |\Omega(n)\rangle = 0 \qquad 2 \leqslant n \leqslant N-2 \tag{13}$$

and is non-local at the boundaries:

$$\mathcal{U}_0 |\Omega(1)\rangle = -\epsilon_s \sum_{n=1}^{N-1} (-\epsilon_s Q)^{n}\ |\Omega(n)\rangle \qquad \mathcal{U}_0 |\Omega(N-1)\rangle = (-\epsilon_s Q)^{-N}\, \mathcal{U}_0 |\Omega(1)\rangle\,. \tag{14}$$

Next, the action of the operator $\mathcal{U} = \sum_{k=1}^{N-1} U_k$ on the states $|\Omega(n)\rangle$ gives the following equations:

$$\begin{aligned}
&\mathcal{U}\,|\Omega(1)\rangle = (Q + Q^{-1})\,|\Omega(1)\rangle + \epsilon_s\,|\Omega(2)\rangle \\
&\mathcal{U}\,|\Omega(n)\rangle = (Q + Q^{-1})\,|\Omega(n)\rangle + \epsilon_s\,|\Omega(n-1)\rangle + \epsilon_s\,|\Omega(n+1)\rangle \qquad \text{for } \; 2 \leqslant n \leqslant N-2 \\
&\mathcal{U}\,|\Omega(N-1)\rangle = (Q + Q^{-1})\,|\Omega(N-1)\rangle + \epsilon_s\,|\Omega(N-2)\rangle\,.
\end{aligned} \tag{15}$$

Before we substitute these results in the eigenvalue equation, we will define two new states

$$\epsilon_s\,|\Omega(0)\rangle = \mathcal{U}_0\,|\Omega(1)\rangle \qquad \epsilon_s\,|\Omega(N)\rangle = \mathcal{U}_0\,|\Omega(N-1)\rangle \tag{16}$$

to include the cases $n = 0$ and $n = N$ in the definition of Ψ_{2s}, equation (8). Finally, the action of $\mathcal{H} = \mathcal{U} + \mathcal{U}_0$ on the states $|\Omega(n)\rangle$ is

$$\begin{aligned}
&\mathcal{H}\,|\Omega(0)\rangle = (Q + Q^{-1})\,|\Omega(0)\rangle + (-\epsilon_s Q)^N \epsilon_s\,|\Omega(N-1)\rangle + \epsilon_s\,|\Omega(1)\rangle \\
&\mathcal{H}\,|\Omega(n)\rangle = (Q + Q^{-1})\,|\Omega(n)\rangle + \epsilon_s\,|\Omega(n-1)\rangle + \epsilon_s\,|\Omega(n+1)\rangle \\
&\qquad\qquad \text{for } \; 1 \leqslant n \leqslant N-2 \\
&\mathcal{H}\,|\Omega(N-1)\rangle = (Q + Q^{-1})\,|\Omega(N-1)\rangle + \epsilon_s\,|\Omega(N-2)\rangle + (-\epsilon_s Q)^{-N} \epsilon_s\,|\Omega(0)\rangle \\
&\mathcal{H}\,|\Omega(N)\rangle = (Q + Q^{-1})\,|\Omega(N)\rangle + \epsilon_s\,|\Omega(N-1)\rangle + (-\epsilon_s Q)^{-N} \epsilon_s\,|\Omega(1)\rangle\,.
\end{aligned} \tag{17}$$

Substituting these results in the eigenvalue equation $\mathcal{H}\Psi_{2s} = E_{2s}\,\Psi_{2s}$ and using the boundary conditions

$$(-\epsilon_s Q)^N A(x) = A(N + x) \tag{18}$$

we obtain a complete set of eigenvalue equations for the wavefunctions:

$$E_{2s}\;A(n) = (Q + Q^{-1})A(n) + \epsilon_s A(n-1) + \epsilon_s A(n+1) \qquad \text{for } \; 1 \leqslant n \leqslant N-1. \tag{19}$$

The plane-wave parametrization $A(n) = \xi^n$ solves these eigenvalue equations and the boundary conditions (18), provided that

$$E_{2s} = Q + Q^{-1} + \epsilon_s(\xi + \xi^{-1}) \qquad \text{and} \qquad \xi^N = (-\epsilon_s Q)^N \tag{20}$$

where $\xi = \mathrm{e}^{\mathrm{i}\theta}$, θ being the momentum.

3.2. Two-pseudoparticle eigenstates

Let us now consider the sector $r = 2(2s)$, where we have two interacting pseudoparticles. We seek the corresponding eigenfunction as products of single pseudoparticles eigenfunctions, i.e.

$$\Psi_{4s} = \sum_{x_1+1<x_2} A(x_1, x_2)\,|\Omega(x_1, x_2)\rangle \tag{21}$$

where

$$|\Omega(x_1, x_2)\rangle = \sum_{i=-s}^{s} \sum_{j=-s}^{s} (-1)^{i+j} q^{2s-i-j}\,|x_1(-i, i), x_2(-j, j)\rangle\,. \tag{22}$$

To solve the eigenvalue equation $\mathcal{H}\Psi_{4s} = E_{4s}\Psi_{4s}$, we recall (7) to obtain the action of $\mathcal{U}$ and $\mathcal{U}_0$ on the states $|\Omega(x_1, x_2)\rangle$. We have to consider four cases.

(i) When the two pseudoparticles are separated in the bulk, the action of $\mathcal{U}$ is

$$\mathcal{U}\,|\Omega(x_1,x_2)\rangle = 2(Q+Q^{-1})\,|\Omega(x_1,x_2)\rangle + \epsilon_s\,|\Omega(x_1-1,x_2)\rangle + \epsilon_s\,|\Omega(x_1+1,x_2)\rangle$$
$$+\epsilon_s\,|\Omega(x_1,x_2-1)\rangle + \epsilon_s\,|\Omega(x_1,x_2+1)\rangle \tag{23}$$

i.e. for $x_1 \geqslant 2$ and $x_1+3 \leqslant x_2 \leqslant N-2$.
(ii) When the two pseudoparticles are separated but one of them or both are at the boundaries

$$\mathcal{U}\,|\Omega(1,x_2)\rangle = 2(Q+Q^{-1})\,|\Omega(1,x_2)\rangle + \epsilon_s\,|\Omega(2,x_2)\rangle + \epsilon_s\,|\Omega(1,x_2-1)\rangle$$
$$+\epsilon_s\,|\Omega(1,x_2+1)\rangle \tag{24}$$

$$\mathcal{U}\,|\Omega(x_1,N-1)\rangle = 2(Q+Q^{-1})\,|\Omega(x_1,N-1)\rangle + \epsilon_s\,|\Omega(x_1-1,N-1)\rangle$$
$$+\epsilon_s\,|\Omega(x_1+1,N-1)\rangle + \epsilon_s\,|\Omega(x_1,N-2)\rangle \tag{25}$$

$$\mathcal{U}\,|\Omega(1,N-1)\rangle = 2(Q+Q^{-1})\,|\Omega(1,N-1)\rangle + \epsilon_s\,|\Omega(2,N-1)\rangle + \epsilon_s\,|\Omega(1,N-2)\rangle \tag{26}$$

where $2 \leqslant x_1 \leqslant N-4$ and $4 \leqslant x_2 \leqslant N-2$.
(iii) When the two pseudoparticles are neighbours in the bulk

$$\mathcal{U}\,|\Omega(x,x+2)\rangle = 2(Q+Q^{-1})\,|\Omega(x,x+2)\rangle + \epsilon_s\,|\Omega(x-1,x+2)\rangle + \epsilon_s\,|\Omega(x,x+3)\rangle$$
$$+U_{x+1}\,|\Omega(x,x+2)\rangle \tag{27}$$

for $2 \leqslant x \leqslant N-4$.
(iv) When the two pseudoparticles are neighbours and at the boundaries

$$\mathcal{U}\,|\Omega(1,3)\rangle = 2(Q+Q^{-1})\,|\Omega(1,3)\rangle + \epsilon_s\,|\Omega(1,4)\rangle + U_2\,|\Omega(1,3)\rangle \tag{28}$$

$$\mathcal{U}\,|\Omega(N-3,N-1)\rangle = 2(Q+Q^{-1})\,|\Omega(N-3,N-1)\rangle + \epsilon_s\,|\Omega(N-4,N-1)\rangle$$
$$+U_{N-2}\,|\Omega(N-3,N-1)\rangle\,. \tag{29}$$

Moreover, the action of $\mathcal{U}_0$ does not depend on the pseudoparticles are neither separated nor neighbours. It vanishes in the bulk

$$\mathcal{U}_0\,|\Omega(x_1,x_2)\rangle = 0 \qquad \text{for} \quad x_1 \neq 1 \quad \text{and} \quad x_2 \neq N-1 \tag{30}$$

and is non-zero at the boundaries:

$$\mathcal{U}_0\,|\Omega(1,x_2)\rangle = -\epsilon_s \sum_{k=1}^{x_2-2}(-\epsilon_s Q)^k\,|\Omega(k,x_2)\rangle - (-\epsilon_s Q)^{x_2-1}U_{x_2}\,|\Omega(x_2-1,x_2+1)\rangle$$
$$-\epsilon_s \sum_{k=x_2+2}^{N-1}(-\epsilon_s Q)^{k-2}\,|\Omega(x_2,k)\rangle \tag{31}$$

$$\mathcal{U}_0\,|\Omega(x_1,N-1)\rangle = (-\epsilon_s Q)^{-N+2}\,\mathcal{U}_0\,|\Omega(1,x_2)\rangle \tag{32}$$

where $2 \leqslant x_1 \leqslant N-3$ and $3 \leqslant x_2 \leqslant N-2$.

Before we substitute these expressions in the eigenvalue equation, we define new states in order to have consistency between bulk and boundaries terms

$$\mathcal{U}_0\,|\Omega(1,x_2)\rangle = \epsilon_s\,|\Omega(0,x_2)\rangle\,, \quad \mathcal{U}_0\,|\Omega(x_1,N-1)\rangle = \epsilon_s\,|\Omega(x_1,N)\rangle$$
$$\mathcal{U}_0\,|\Omega(1,N-1)\rangle = \epsilon_s\,|\Omega(0,N-1)\rangle + \epsilon_s\,|\Omega(1,N)\rangle \tag{33}$$
$$U_{x+1}\,|\Omega(x,x+2)\rangle = \epsilon_s\,|\Omega(x,x+1)\rangle + \epsilon_s\,|\Omega(x+1,x+2)\rangle\,.$$

Acting with $\mathcal{H}$ on these new states, we obtain

$$\mathcal{H}\,|\Omega(0,x_2)\rangle = 2(Q+Q^{-1})\,|\Omega(0,x_2)\rangle + \epsilon_s\,|\Omega(0,x_2-1)\rangle + \epsilon_s\,|\Omega(0,x_2+1)\rangle$$
$$+\epsilon_s\,|\Omega(1,x_2)\rangle + (-\epsilon_s Q)^{N-2}\epsilon_s\,|\Omega(x_2,N-1)\rangle \tag{34}$$

$$\mathcal{H}\,|\Omega(x_1,N)\rangle = 2(Q+Q^{-1})\,|\Omega(x_1,N)\rangle + \epsilon_s\,|\Omega(x_1-1,N)\rangle + \epsilon_s\,|\Omega(x_1+1,N)\rangle$$
$$+\epsilon_s\,|\Omega(x_1,N-1)\rangle + (-\epsilon_s Q)^{-N+2}\epsilon_s\,|\Omega(1,x_1)\rangle \tag{35}$$

$$\mathcal{H}\,|\Omega(x,x+1)\rangle = (Q+Q^{-1})\,|\Omega(x,x+1)\rangle + \epsilon_s\,|\Omega(x-1,x+1)\rangle + \epsilon_s\,|\Omega(x,x+2)\rangle\,. \tag{36}$$

Substituting these results in the eigenvalue equation, we obtain the following equations for wavefunctions corresponding to the separated pseudoparticles:

$$E_{4s}A(x_1,x_2) = 2(Q+Q^{-1})A(x_1,x_2) + \epsilon_s A(x_1-1,x_2) + \epsilon_s A(x_1+1,x_2)$$
$$+\epsilon_s A(x_1,x_2-1) + \epsilon_s A(x_1,x_2+1) \tag{37}$$

i.e. for $x_1 \geqslant 1$ and $x_1+3 \leqslant x_2 \leqslant N-1$. The boundary conditions now read

$$A(x_2, N+x_1) = (-\epsilon_s Q)^{N-2}A(x_1,x_2). \tag{38}$$

The parametrization for the wavefunctions

$$A(x_1,x_2) = A_{12}\xi_1^{x_1}\xi_2^{x_2} + A_{21}\xi_1^{x_2}\xi_2^{x_1} \tag{39}$$

solves equation (37) provided that

$$E_{4s} = 2(Q+Q^{-1}) + \epsilon_s(\xi_1+\xi_1^{-1}+\xi_2+\xi_2^{-1}) \tag{40}$$

and the boundary conditions (38) provided that

$$\xi_2^N = (-\epsilon_s Q)^{N-2}\frac{A_{21}}{A_{12}}, \quad \xi_1^N = (-\epsilon_s Q)^{N-2}\frac{A_{12}}{A_{21}} \quad \Rightarrow \quad \xi^N = (-\epsilon_s Q)^{2(N-2)} \tag{41}$$

where $\xi = \xi_1\xi_2 = \mathrm{e}^{\mathrm{i}(\theta_1+\theta_2)}$, $\theta_1+\theta_2$ being the total momenta.

Now we include the new states (33) in the definition of Ψ_{4s} in order to extend (21) to

$$\Psi_{4s} = \sum_{x_1<x_2} A(x_1,x_2)\,|\Omega(x_1,x_2)\rangle\,. \tag{42}$$

Here we have used the same notation for separate and neighbouring states.

Substituting equations (27) and (36) in the eigenvalue equation, we obtain

$$E_{4s}A(x,x+1) = (Q+Q^{-1})A(x,x+1) + \epsilon_s A(x-1,x+1) + \epsilon_s A(x,x+2) \tag{43}$$

which gives us the phase shift produced by the interchange of the two interacting pseudoparticles

$$\frac{A_{21}}{A_{12}} = -\frac{1+\xi+\epsilon_s(Q+Q^{-1})\xi_2}{1+\xi+\epsilon_s(Q+Q^{-1})\xi_1}. \tag{44}$$

We thus arrive at the Bethe ansatz equations which fix the values of ξ_1 and ξ_2:

$$\xi_2^N = (-\epsilon_s Q)^{N-2}\left\{-\frac{1+\xi+\epsilon_s(Q+Q^{-1})\xi_2}{1+\xi+\epsilon_s(Q+Q^{-1})\xi_1}\right\}$$
$$\xi_1^N\xi_2^N = (-\epsilon_s Q)^{2(N-2)} \tag{45}$$

3.3. General eigenstates

The generalization to any r multiple of $2s$ is in principle straightforward. Since the Yang–Baxter equations are satisfied, there is only two-pseudoparticle scattering (if we use S-matrix language). Therefore the neighbour equations, where more the two pseudoparticles become neighbours, are not expected to give any new restrictions. For instance, in the sector $r = 3(2s)$ we have three interacting pseudoparticles with parameters ξ_1, ξ_2 and ξ_3. The corresponding wavefunctions

$$A(x_1, x_2, x_3) = A_{123}\xi_1^{x_1}\xi_2^{x_2}\xi_3^{x_3} + A_{132}\xi_1^{x_1}\xi_2^{x_3}\xi_3^{x_2} + A_{213}\xi_1^{x_2}\xi_2^{x1}\xi_3^{x_3} + A_{231}\xi_1^{x_2}\xi_2^{x_3}\xi_3^{x_1}$$
$$+ A_{312}\xi_1^{x_3}\xi_2^{x_1}\xi_3^{x_2} + A_{321}\xi_1^{x_3}\xi_2^{x_2}\xi_3^{x_1} \tag{46}$$

satisfy the boundary conditions

$$A(x_2, x_3, N + x_1) = (-\epsilon_s Q)^{N-4} A(x_1, x_2, x_3)$$

which imply that

$$\xi_i^N = (-\epsilon_s Q)^{N-4}\frac{A_{ijk}}{A_{jki}} = (-\epsilon_s Q)^{N-4}\frac{A_{ikj}}{A_{kji}} \qquad i \neq j \neq k = 1, 2, 3. \tag{47}$$

These relations show us that the interchange of two pseudoparticles is independent of the position of the third particle. Thus in the sector $r = p(2s)$, we expect that the p-pseudoparticle phase shift will be a sum of $\binom{p}{2}$ two-pseudoparticle phase shifts and the energy is given by

$$E_{p(2s)} = \sum_{n=1}^{p} \left\{ Q + Q^{-1} + \epsilon_s(\xi_n + \xi_n^{-1}) \right\} \tag{48}$$

where

$$\xi_a^N = (-\epsilon_s Q)^{N-2p+2} \prod_{b\neq a}^{p} \left\{ -\frac{1 + \xi_a\xi_b + \epsilon_s(Q + Q^{-1})\xi_a}{1 + \xi_a\xi_b + \epsilon_s(Q + Q^{-1})\xi_b} \right\} \qquad a = 1, \ldots, p \tag{49}$$

$$\left(\xi_1\xi_2 \cdots \xi_p\right)^N = (-\epsilon_s Q)^{p(N-2p+2)}.$$

This is not all; in a sector r we may have p pseudoparticle and $N_{s-1}, N_{s-2}, \ldots, N_{-s+1}$ impurities of the type $(s-1), (s-2), \ldots, (-s+1)$, respectively, such that

$$N_{s-1} + 2N_{s-2} + \cdots + (2s-1)N_{-s+1} = r - 2sp. \tag{50}$$

We call a state $|a, n\rangle$ flanked by at least two states $|b, n \pm 1\rangle$ such that $a + b \neq 0$ an impurity. Since $\mathcal{H}$ is a sum of projectors on spin zero, these states are annihilated by $\mathcal{H}$. In particular, the do not move under the action of $\mathcal{H}$, which is the reason for their name. Nevertheless, a pseudoparticle can propagate past the isolated impurity, but in so doing causes a shift in its position by two lattice sites. Thus, for a sector r with l impurities with parameters $\xi_1, \ldots, \xi_l$ and p pseudoparticles with parameters $\xi_{l+1}, \ldots, \xi_{l+p}$ the energy is given by (48), and the Bethe equations do not depend on impurity type and are given by

$$\xi_a^N \xi_1^2\xi_2^2 \cdots \xi_l^2 = (-\epsilon_s Q)^{N-2p+2} \prod_{\substack{b=l+1 \\ b\neq a}}^{l+p} \left\{ -\frac{1 + \xi_a\xi_b + \epsilon_s(Q + Q^{-1})\xi_a}{1 + \xi_a\xi_b + \epsilon_s(Q + Q^{-1})\xi_b} \right\} \tag{51}$$

with $a = l+1, l+2, \ldots, l+p,\ p \geqslant 1$, and

$$\xi^{2p}(\xi_{l+1} \cdots \xi_{l+p})^{N-2p} = (-\epsilon_s Q)^{p(N-2p+2)} \tag{52}$$

where $\xi = \xi_1\xi_2 \cdots \xi_l\xi_{l+1} \cdots \xi_{l+p}$.

4. Conclusion

We have shown that these closed Temperley–Lieb quantum-invariant spin chains can be solved by the coordinate Bethe ansatz. A consequence of the non-local terms $\mathcal{U}_0$ is that boundary conditions arise which depend on the quantum group parameter q via the relation $Q + Q^{-1} = [2s + 1]_q$ and on the number p of pseudoparticles (which is equal to the spin sector r, when $s = \frac{1}{2}$).

An interesting extension of this work would be the application of the methods presented here to solving new strongly correlated electronic systems associated with the Temperley–Lieb algebras [20, 21]. This is under investigation at present.

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